Sets, Relations & Functions

Equivalence Relations

A → non-empty set
R → Relation on A

Defⁿ

[Note. Let A & B be two sets. A relation from A to B is a subset of $A \times B$. We denote a relation by R.
$\therefore$ R is a relation from A to B if $R \subseteq A \times B$]

The relation R is said to be equivalent relation if it is —

1. Reflexive i.e. $\forall a \in A$, aRa
2. Symmetric i.e. if aRb then bRa
$aRb \Leftrightarrow bRa$
3. Transitive i.e.
aRb & $bRc \Rightarrow aRc$

e.g.
1. Equality of numbers on set of real numbers
2. Congruency of triangles on the set of triangles.

Partially ordered Sets (Poset)

A relation R on a set A is said to be partial order if R is reflexive, anti-symmetric and transitive. ∴ a relation R on a set A is partial order relation if -

1. R is reflexive i.e.
   $aRa \ \forall a \in A$
   This is called Reflexivity
2. R is anti-symmetric i.e.
   If aRb & bRa then $a = b$
3. R follows transitivity i.e.
   If aRb, bRc then aRc

Then the set A with partial order R is called Partial Ordered Set (Poset)

We denote R by $\preceq$
& ordered pair $(A, \preceq) \rightarrow$ Poset

Ex The relation of divisibility is a partial order on the set of Natural numbers N. i.e. $(N, \preceq)$ is a poset where $a \preceq b$ means $a \mid b$

Ex The relation $<$ on N is not partial order as it is not reflexive.

Def^n. Let $(A, \leq)$ be a poset. The elements a & b of A are said to be comparable if $a \leq b$ or $b \leq a$.

∴ a & b are non-comparable if neither $a \leq b$ nor $b \leq a$.

e.g. Consider $(N, \leq)$

'$\leq$' is divisibility.

Then 2 & 5 are not comparable as 2 does not divide 5 & vice versa.

Thus in a poset every pair of elements need not be comparable.

Def^n $(A, \leq) \rightarrow$ poset

We say $a < b$ if $a \leq b$ but $a \neq b$.

We also say 'b' is larger than or equal to 'a' & write $b \geq a$ if $a \leq b$.

Totally ordered Set (Chain) (Linearly ordered Set)

$(A, \leq) \rightarrow$ poset

It is called totally ordered set or chain if every two elements in A are comparable i.e.

if $a, b \in A$ then either $a \leq b$ or $b \leq a$

e.g $(R, \leq)$ is a chain

$\leq \rightarrow$ less than or equal to

Hasse Diagram

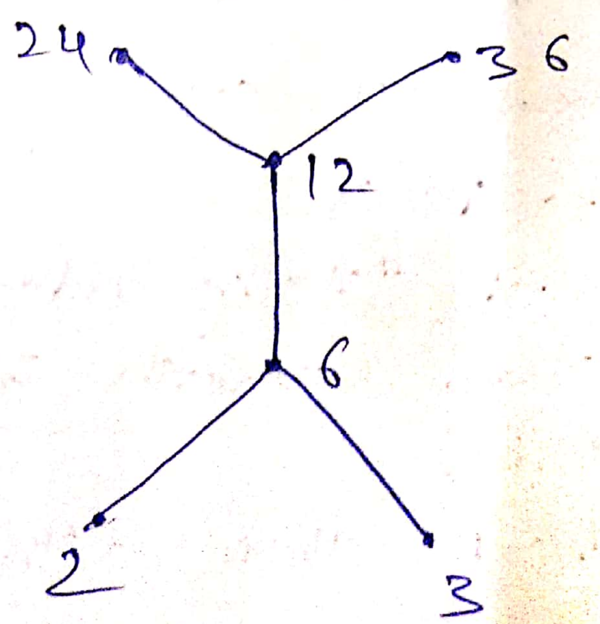
A partial ordering ≤ on a poset say P can be represented by Hasse Diagram. In this diagram, each element is represented by a small circle or by a dot and any two comparable elements are joined in such a way that if $a \leq b$ then a lies below b.

Non-comparable elements are not joined. ∴ there will not be any horizontal lines in Hasse diagram.

Ex:- Let $X = \{2, 3, 6, 12, 24, 36\}$

≤ → $x \leq y$ if x divides y

Then Hasse diagram of $(X, \leq)$ is

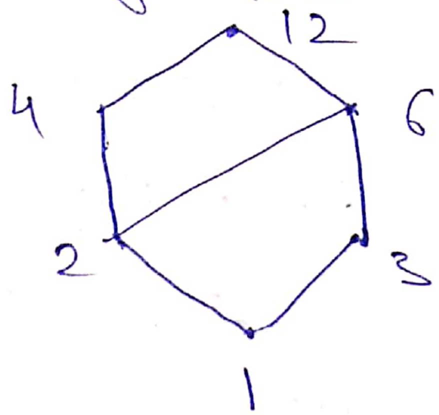


Ex. $X \rightarrow$ set of factors of 12

$X = \{1, 2, 3, 4, 6, 12\}$

$x \leq y$ iff x divides y.

Hasse Diagram



Maximal & minimal Elements

$(P, \leq) \rightarrow$ Poset

An element $a \in P$ is called a minimal element if there is no other element b in P s.t. $b \leq a$ with $b \neq a$.

A minimal element need not be unique. All those elements which are at the lowest level of a Hasse diagram of a poset are minimal elements

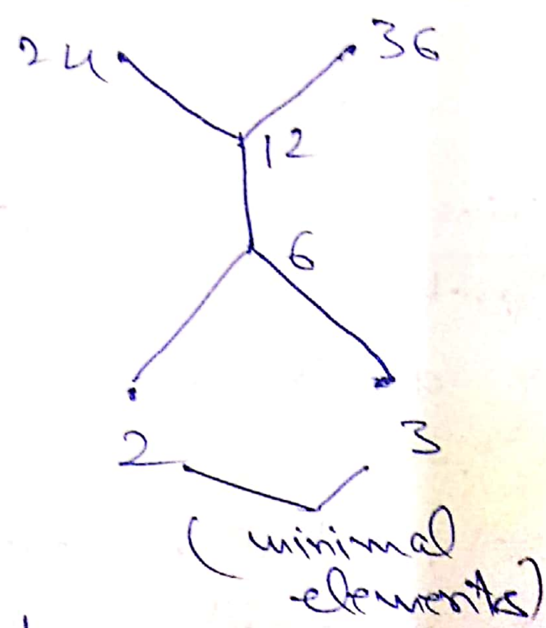
An element $a \in P$ is called a maximal element if there is no b in P s.t. $a \leq b$ with $a \neq b$.

- Need not be unique
- at the highest level of a Hasse Diagram.

Note- A poset may or may not have minimal / maximal element

Note- Every finite poset has at least one minimal element

Ex- $X = \{2, 3, 6, 12, 24, 36\}$, / divisibility



24, 36 → maximal elements

Least Element

$(P, \leq) \rightarrow$ Poset

If $\exists$ an element $a \in P$ s.t. $a \leq x, \forall x \in P$ then 'a' is called the least element in P.

— denote it by '0'

— zero element

Ex $X = \{1, 2, 3, 4, 6, 12\}$

$x \leq y$ iff x divides $y \Rightarrow (X, \leq)$ - Poset

$1 \rightarrow$ least element